

Fixed Point Result on Generalized Cone b-Metric Spaces

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Abstract: The purpose of this paper is to prove a fixed point result on contraction mapping in generalized cone b-metric space (in short GCbMS) as a generalization of cone metric space, cone b metric space and rectangular metric space. The conception of generalized metric space is a generalization of that of classical metric space. Several authors have proved fixed point theorems of contractive mappings on generalized metric spaces, which also generalized some corresponding fixed point results in classical metric spaces. In present paper, we prove a result that is extension of the Kannan fixed point theorem proved by Reny George *et al.* Our result is extend and unify several well known results in the literature available for cone and cone-b metric space.

Keywords: Metric Space, b-metric Space, Cone Metric Space, Cone b-metric Space, Rectangular Metric Space

1. Introduction

Fixed point theorems have wide applications in different fields of mathematics. Due to which, existence as well as uniqueness of fixed points and common fixed points has turn into a subject of great importance. This is due to verify Banach Contraction Principle in different directions. In the recent four to five decades many authors generalized the Banach contraction Principle by moderating the triangular inequality of a metric space. generalized metric space [2, 7, 9-10, 16, 25], cone metric space [11], b metric space [4, 5, 6, 8 references therein], cone b metric space [11, 12, 13, 16-22], rectangular metric space [21], cone rectangular metric space [14, 19, 20], are some of the generalizations of metric space introduced by different authors in past few decades. Analogue Banach contraction principle, Kannan contraction principle, Ciric contraction principle and lots of the existing fixed-point theorems for various generalized contractions were proved in these generalized spaces.

Most of the generalization of metric space are Hausdorff topology but we can also find generalization of metric space which are not necessarily Hausdorff topology [15, 21, 24, 25]. Tarskian mathematician used non Hausdorff topology to programming language semantics used in computer science.

Here, we prove a fixed point theorem for contraction mapping in generalized cone b-metric space (in short GCbMS) as a generalization of cone metric space, cone b metric space and rectangular metric space which is the extension of Kannan fixed point theorem proved by Reny George *et al.* [17].

2. Preliminaries

(1) Let E be a real Banach spaces and $P \subset E$. P is called a cone iff

- (i) P is closed and non-empty and $P \neq \{\theta\}$
- (ii) $ax + by \in P \forall x, y \in P$ and a, b are non negative real
- (iii) if $x \in P; -x \in P \Rightarrow x = 0$ i.e. $P \cap -P = \{\theta\}$

Given a cone $P \subset E$ we define a partial ordering \preceq with respect to P as

- (i) $x \preceq y$ if and only if $y - x \in P$
- (ii) $x \preceq y$ & $x \neq y$, then we may write $x < y$
- (iii) $x \ll y \Rightarrow y - x \in \text{int } P$ and
- (iv) if $\text{int } P \neq \emptyset$, cone P is called solid cone.
- (v) The cone P is said to be Normal if there exist a number $k > 0 \forall x, y \in E$

$$0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$$

The least positive number satisfying above is called the normal constant of P .

(vi) The cone is said to be regular if every increasing sequence $\{x_n\}$ which is bounded from above is convergent i.e. if $\{x_n\}$ is a sequence $x_1 \leq x_2 \leq x_3 \leq \dots x_n \leq \dots \leq y$ some $y \in E \exists x \in E$ such that $\lim_{n \rightarrow \infty} x_n = 0$.

Equivalently the cone is said to be regular if every increasing sequence $\{x_n\}$ which is bounded from below is convergent i.e. if $\{x_n\}$ is a sequence $x_1 \geq x_2 \geq x_3 \geq \dots x_n \geq \dots \geq y$ some $y \in E \exists x \in E$, such that $\lim_{n \rightarrow \infty} x_n = 0$.

Note: A regular cone is a normal cone.

(2) Definition: Let E be a topological vector space over the field $F = R$ or C and let $P \subseteq E$ be a closed set with following axioms:

(C1) $P \neq \emptyset, \{0\}$

(C2) $x, y \in P \& \delta \geq 0 \Rightarrow x + y, \delta x \in P$

(C3) $P \cap -P = \{0\}$,

Then P is called a cone. If, in addition, $\text{int } P \neq \emptyset$, we say that P is a solid cone.

(3) Definition: Let X be a non-empty set and $\varphi: X \rightarrow X$ is a mapping such that $\forall x, y, z \in X$ satisfies

(CMS1) $0 \leq \varphi(x, y) \forall x, y \& x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CMS2) $\varphi(x, y) = \varphi(y, x) \forall x, y \& x \neq y$

(CMS3) $\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y)$

Then φ is called a cone metric on X and (X, φ) is called cone metric space.

(4) Definition: The pair (X, d) is called b-metric space with coefficient $s \geq 1$ such that X is a non empty set and a mapping $d: X \times X \rightarrow R^+$ satisfies the following axioms for all $x, y, z \in X$

(bMS₁) $d(x, y) = 0$ if and only if $x = y$

$$\varphi(x, y) = (d(x, y))^p \leq (d(x, z) + d(z, y))^p \leq 2^{p-1}(d(x, z))^p + (d(z, y))^p \leq 2^{p-1}\{\varphi(x, z) + \varphi(z, y)\}$$

Hence (X, φ) is a Cone b-metric space.

(9) Definition: Let X be a non-empty set and $\varphi: X \rightarrow X$ is a mapping such that $\forall x, y, z, a, b \in X$ satisfies

(CRM1) $0 \leq \varphi(x, y) \forall x, y \& x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CRM2) $\varphi(x, y) = \varphi(y, x) \forall x, y \& x \neq y$

(CRM3) $\varphi(x, y) \leq \varphi(x, a) + \varphi(a, b) + \varphi(b, y)$

Then φ is called a Cone Rectangular Metric on X and (X, φ) is called Cone Rectangular Metric Space.

(10) Definition: Let X be a non empty set, $s \geq 1$ is a real and $\varphi: X \rightarrow X$ is a mapping such that $\forall x, y, z, a, b \in X$ satisfies

(GCRMS1) $0 \leq \varphi(x, y) \forall x, y \& x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

$$d(3, 2) = (12, 12) > d(3, 4) + d(4, 5) + d(5, 2) = (2, 2) + (4, 4) + (4, 4) = (10, 10)$$

MAIN RESULTS: Reny George *et. al.* [17] proved Banach contraction principle and Kannan contraction principle. We extended their results in the present paper.

Theorem 3: Let (X, φ) be a complete generalized cone b-

$$(bMS_2) d(x, y) = d(y, x)$$

$$(bMS_3) d(x, y) \leq s\{d(x, z) + d(z, y)\}$$

(5) Example: Let $X = R$ be the set of real number and $d(x, y) = |x - y|$ a usual metric, then

$\varphi(x, y) = |x - y|$ is a b-metric space for $k = 2$ but not for R .

(6) Example: The space l_p ($0 < p < 1$)

$$l_p = \{(x_n \in R: \sum_{n=1}^{\infty} |x_n|^p < \infty)\}$$

Together with the mapping $d: l_p \times l_p \rightarrow R$

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

is a b-metric spaces.

(7) Definition: Let X be a non-empty set, $s \geq 1$ is a real and $\varphi: X \rightarrow X$ is a mapping such that $\forall x, y, z \in X$ satisfies

(CBM1) $0 \leq \varphi(x, y) \forall x, y \& x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CBM2) $\varphi(x, y) = \varphi(y, x) \forall x, y \& x \neq y$

(CBM3) $\varphi(x, y) \leq s[\varphi(x, z) + \varphi(z, y)]$

Then φ is called a Cone b-Metric on X and (X, φ) is called Cone b- Metric Space.

(8) Example: Let (X, φ) be a metric space and $\varphi(x, y) = (d(x, y))^p$ where $p > 1$, Then (X, φ) is a Cone b-metric space for $s = 2^{p-1}$.

Proof: (i) if $x = y$ then $\varphi(x, x) = (d(x, x))^p = 0$

$$(ii) \varphi(x, y) = (d(x, y))^p = (d(y, x))^p = \varphi(y, x)$$

(iii) if $0 < p < 1$ then by convexity of the function $f(x) = x^p \Rightarrow (\frac{a+b}{2})^p \leq \frac{1}{2}(a^p + b^p) \Rightarrow (a+b)^p = 2^{p-1}(a^p + b^p)$. So for $x, y, z \in X$, we have

$$(GCRMS2) \varphi(x, y) = \varphi(y, x) \forall x, y \& x \neq y$$

$$(GCRMS3) \varphi(x, y) \leq s[\varphi(x, a) + \varphi(a, b) + \varphi(b, y)]$$

Then φ is called a Generalized Cone Rectangular Metric on X and (X, φ) is called Generalized Cone.

(11) Example: Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\}, X = R = A \cup B, d: X \times X \rightarrow E, \forall x, y \in R$

$d(x, y) = 0$ if $x = y$ and $d(x, y) = d(y, x)$ such that

$$d(x, y) = \begin{cases} (12, 12) & \text{for } x = 3, y = 2 \\ (2, 2) & \text{for } x \in \{2, 3\} \text{ and } y = 4 \\ (4, 4) & \text{for } x \in \{2, 3, 4\} \text{ and } y = 5 \\ (2, 3) & \text{for } x \text{ or } y \notin \{2, 3, 4, 5\} \text{ and } x \neq y \end{cases}$$

Then (X, d) is a GCbMS but not a CRMS as we have

metric space with constant coefficient $s \geq 1$ and let $\alpha_i \geq 0 (i = 1, 2, 3, 4)$, P is a solid cone and $T: X \times X \rightarrow X$ be a mapping satisfying;

$$\varphi(Tx, Ty) \leq \alpha_1 \varphi(x, y) + \alpha_2 \frac{[\varphi(x, Tx) + \varphi(y, Ty)]}{2} + \alpha_3 \frac{[\varphi(x, y) + \varphi(Tx, Ty)]}{2} + \alpha_4 \frac{[\varphi(x, Ty) + \varphi(y, Tx)]}{2s} \quad (1)$$

for all $x, y \in X$ where $s\alpha_1 + \frac{(1+s)}{2}(\alpha_2 + \alpha_3 + \alpha_4) \leq 1$. Then T has a unique fixed point in X .

Proof. Let $x_0 \in X$ be a arbitrary point. Define a sequence $\{x_n\} \in X$ such that $Tx_n = x_{n+1}$ ($n \geq 0$). Now we are to show that sequence $\{x_n\}$ is a Cauchy sequence. If for any

$n; x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T . Therefore, there is no need to go further. Otherwise $x_{n+1} \neq x_n \forall n \geq 1$; setting $\varphi_n = \varphi(x_n, x_{n+1})$. Thus inserting $x_{n-1}; x_n$ for $x; y$ respectively in inequality (3.1), we have

$$\begin{aligned} \alpha_1 \varphi(x_{n-1}, x_n) + \alpha_2 \frac{[\varphi(x_{n-1}, Tx_{n-1}) + \varphi(x_n, Tx_n)]}{2} + \alpha_3 \frac{[\varphi(x_{n-1}, x_n) + \varphi(Tx_{n-1}, Tx_n)]}{2} + \alpha_4 \frac{[\varphi(x_{n-1}, Tx_n) + \varphi(x_n, Tx_{n-1})]}{2s} \\ \varphi(x_n, x_{n+1}) \leq \alpha_1 \varphi(x_{n-1}, x_n) + \alpha_2 \frac{[\varphi(x_{n-1}, x_n) + \varphi(x_n, x_{n+1})]}{2} + \alpha_3 \frac{[\varphi(x_{n-1}, x_n) + \varphi(x_n, x_{n+1})]}{2} + \alpha_4 \frac{[\varphi(x_{n-1}, x_{n+1}) + \varphi(x_n, x_n)]}{2s} \\ \varphi_n \leq \alpha_1 \varphi_{n-1} + \alpha_2 \frac{[\varphi_{n-1} + \varphi_n]}{2} + \alpha_3 \frac{[\varphi_{n-1} + \varphi_n]}{2} + \alpha_4 \frac{s[\varphi_{n-1} + \varphi_n]}{2s} \\ \left(1 - \frac{\alpha_2}{2} - \frac{\alpha_3}{2} - \frac{\alpha_4}{2}\right) \varphi_n \leq \left(\alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2}\right) \varphi_{n-1} \\ \varphi_n \leq \frac{(\alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2})}{(1 - \frac{\alpha_2}{2} - \frac{\alpha_3}{2} - \frac{\alpha_4}{2})} \varphi_{n-1} \\ \varphi_n \leq \gamma \varphi_{n-1} \end{aligned}$$

Where

$$\gamma = \frac{(\alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2})}{(1 - \frac{\alpha_2}{2} - \frac{\alpha_3}{2} - \frac{\alpha_4}{2})} \leq \frac{1}{s} \Rightarrow s\alpha_1 + \frac{(1+s)}{2}(\alpha_2 + \alpha_3 + \alpha_4) \leq 1.$$

Repeating iteration n times; we have

$$\varphi_n \leq \gamma^n \varphi_0 \quad (2)$$

Assume; x_0 is not a periodic point of φ . Indeed, if $x_n = x_0$ then; from (3.2), for any $n \geq 2$; we conclude that

$$\varphi_0 = \varphi(x_0, x_1) = \varphi(x_0, Tx_0) = \varphi(x_n, Tx_n) = \varphi(x_n, x_{n+1}) = \varphi_n \leq \gamma^n \varphi_0$$

Since $\gamma \in [0, 1)$; we obtain $-\varphi_0 \in P \Rightarrow \varphi_0 = 0 \Rightarrow x_0 = x_1$ i.e. x_0 is a fixed point for T . Therefore we assume that $x_n \neq x_m \forall n, m \in N$ and $n \neq m$, again letting $\varphi_n^* = \varphi(x_n, x_{n+2})$ and using (3.1), we observe that

$$\varphi_n^* = \varphi(x_n, x_{n+2}) = \varphi(Tx_{n-1}, Tx_{n+1}) \leq \gamma \varphi(x_{n-1}, x_{n+1}) \leq \gamma \varphi_{n-1}^*$$

Repeating iteration n times; we have $\varphi_n^* \leq \gamma^n \varphi_0^*$

Now for any sequence $\{x_n\}$, two cases may be arising for $\varphi(x_n, x_{n+p})$

(1) $p = 2m + 1$ i.e. p is odd. Letting $n \geq 2$ and $m > n$; we have

$$\begin{aligned} \varphi(x_n, x_{n+2m+1}) &\leq s [\varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, x_{n+2}) + \varphi(x_{n+2}, x_{n+2m+1})] \\ &\varphi(x_n, x_{n+2m+1}) \leq s(\varphi_n + \varphi_{n+1}) \\ &+ s^2 [\varphi(x_{n+2}, x_{n+3}) + \varphi(x_{n+3}, x_{n+4}) + \varphi(x_{n+4}, x_{n+2m+1})] \\ \varphi(x_n, x_{n+2m+1}) &\leq s(\varphi_n + \varphi_{n+1}) + s^2[(\varphi_{n+2} + \varphi_{n+3})] \\ &+ s^2 [\varphi(x_{n+4}, x_{n+5}) + \varphi(x_{n+5}, x_{n+2m+1})] \\ \varphi(x_n, x_{n+2m+1}) &\leq s(\varphi_n + \varphi_{n+1}) + s^2[(\varphi_{n+2} + \varphi_{n+3})] + s^3[(\varphi_{n+4} + \varphi_{n+5})] + \dots \\ &+ s^{m-1}[\varphi(\varphi_{n+2m-2} + \varphi_{n+2m-1})] + s^m \varphi_{n+2m} \\ \varphi(x_n, x_{n+2m+1}) &\leq s(\gamma^n \varphi_0 + \gamma^{n+1} \varphi_0) + s^2(\gamma^{n+2} \varphi_0 + \gamma^{n+3} \varphi_0) \\ &+ s^3(\gamma^{n+4} \varphi_0 + \gamma^{n+5} \varphi_0) + \dots + s^{m-1}(\gamma^{n+2m-2} \varphi_0 + \gamma^{n+2m-1} \varphi_0) + s^m \gamma^{n+2m} \varphi_0 \end{aligned}$$

$$\begin{aligned}
\varphi(x_n, x_{n+2m+1}) &\leq (s\gamma^n + s^2\gamma^{n+2} + \dots)\varphi_0 + (s\gamma^{n+1} + s^2\gamma^{n+3} + \dots)\varphi_0 \\
\varphi(x_n, x_{n+2m+1}) &\leq s\gamma^n(1 + s\gamma^2 + s^2\gamma^4 + \dots)\varphi_0 + s\gamma^{n+1}(1 + s\gamma^2 + s^2\gamma^4 + \dots)\varphi_0 \\
\varphi(x_n, x_{n+2m+1}) &\leq s\gamma^n \frac{1 + \gamma}{1 - s\gamma^2} \varphi_0
\end{aligned}$$

Let $0 \ll c$ (given). Now choose a natural N_π such that $c + N_\pi(0) \subseteq P$, here $N_\pi = \{y \in E, \|y\| < \pi\}$. Now choose another natural N_{π_1} such that $s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \in N_\pi(0) \forall n \in N_{\pi_1}$, so $s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \ll c \forall n \in N_{\pi_1}$, implies $\varphi(x_n, x_{n+2m+1}) \leq s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \ll c \forall n \in N_{\pi_1}$.

(2) $p = 2m$ i.e., p is even. Letting $n \geq 2$ and $m > n$; we have

$$\begin{aligned}
\varphi(x_n, x_{n+2m}) &\leq s[\varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, x_{n+2}) + \varphi(x_{n+2}, x_{n+2m})] \\
\varphi(x_n, x_{n+2m+1}) &\leq s(\varphi_n + \varphi_{n+1}) \\
&+ s^2[\varphi(x_{n+2}, x_{n+3}) + \varphi(x_{n+3}, x_{n+4}) + \varphi(x_{n+4}, x_{n+2m})] \\
\varphi(x_n, x_{n+2m+1}) &\leq s(\varphi_n + \varphi_{n+1}) + s^2[(\varphi_{n+2} + \varphi_{n+3})] \\
&+ s^2[\varphi(x_{n+4}, x_{n+5}) + \varphi(x_{n+5}, x_{n+2m+1})] \\
\varphi(x_n, x_{n+2m+1}) &\leq s[(\varphi_n + \varphi_{n+1})] + s^2[(\varphi_{n+2} + \varphi_{n+3})] \\
&+ s^3[(\varphi_{n+4} + \varphi_{n+5})] + \dots + s^{m-1}(\varphi_{n+2m-2} + \varphi_{n+2m}) \\
\varphi(x_n, x_{n+2m+1}) &\leq s(\gamma^n \varphi_0 + \gamma^{n+1} \varphi_0) + s^2(\gamma^{n+2} \varphi_0 + \gamma^{n+3} \varphi_0) \\
&+ s^3(\gamma^{n+4} \varphi_0 + \gamma^{n+5} \varphi_0) + \dots + s^{m-1}(\gamma^{n+2m-2} \varphi_0 + \gamma^{n+2m-1} \varphi_0) + s^{m-1} \gamma^{n+2m-2} \varphi_0^* \\
\varphi(x_n, x_{n+2m+1}) &\leq (s\gamma^n + s^2\gamma^{n+2} + \dots)\varphi_0 + (s\gamma^{n+1} + s^2\gamma^{n+3} + \dots)\varphi_0 \\
\varphi(x_n, x_{n+2m+1}) &\leq s\gamma^n(1 + s\gamma^2 + s^2\gamma^4 + \dots)\varphi_0 \\
&+ s\gamma^{n+1}(1 + s\gamma^2 + s^2\gamma^4 + \dots)\varphi_0 + s^{m-1} \gamma^{n+2m-2} \varphi_0^* \\
\varphi(x_n, x_{n+2m+1}) &< s\gamma^n \frac{1 + \gamma}{1 - s\gamma^2} \varphi_0 + (s\gamma)^{2m} \gamma^{n-2} \varphi_0^* \\
\varphi(x_n, x_{n+2m+1}) &\leq s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 + \gamma^{n-2} \varphi_0^* \text{ since } \gamma < \frac{1}{s}
\end{aligned}$$

Let $0 \ll c$ (given). Now choose a natural N_{π_2} such that $c + N_{\pi_2}(0) \subseteq P$, here $N_{\pi_2} = \{y \in E, \|y\| < \pi\}$. Now choose another natural N_{π_3} such that $s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \in N_\pi(0) \forall n \in N_{\pi_2}$, so

$$s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \ll c \forall n \in N_{\pi_2}, \text{ implies } \varphi(x_n, x_{n+2m+1}) \leq s\gamma^n \frac{1+\gamma}{1-s\gamma^2} \varphi_0 \ll c \forall n \in N_{\pi_2}. \text{ Let}$$

$$N_\pi = \text{Max}\{N_{\pi_1}; N_{\pi_2}\} \forall n \geq N_\pi \Rightarrow \log_{n \rightarrow \infty} \varphi(x_n, x_{n+p}) \ll c, \text{ implies } \{x_n\} \text{ is a Cauchy sequence.}$$

Therefore; due to completeness of metric spaces (X, φ) , there exist an element $a \in X$ such that $\lim_{n \rightarrow \infty} x_n = a$. Now we shall show that a is a fixed point for T . For that assume $n \in N$, we have

$$\begin{aligned}
\varphi(a, Ta) &\leq s[\varphi(a, x_n) + \varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, Ta)] \\
\varphi(a, Ta) &\leq s[\varphi(a, x_n) + \varphi_n + \varphi(Tx_n, Ta)] \\
\varphi(a, Ta) &\leq s[\varphi(a, x_n) + \gamma^n \varphi_0 + \gamma \varphi(x_n, a)] \\
\varphi(a, Ta) &\leq s[\varphi(x_n, a) + \gamma^n \varphi_0 + \gamma \varphi(x_n, a)] \text{ from (3.2)} \\
\varphi(a, Ta) &\leq s[(1 + \gamma)\varphi(x_n, a) + \gamma^n \varphi_0]
\end{aligned}$$

Now choose a natural $N_{\pi_3}; N_{\pi_4}$ such that $\varphi(x_n, a) \ll \frac{c}{2s(1+\gamma)} \forall n \in N_{\pi_3}$ and $\gamma^n \varphi_0 \ll \frac{c}{2s} \forall n \in N_{\pi_4}$. Let $N_\pi = \text{Max}\{N_{\pi_1}; N_{\pi_2}\} \forall n \geq N_\pi$ so that $\varphi(a, Ta) \leq c \Rightarrow \varphi(a, Ta) = 0 \Rightarrow a = Ta$ implies a is a fixed point.

For uniqueness; assume b is another fixed point of T . Therefore from (3.1), we have

$$\begin{aligned}
\varphi(a, b) &= \varphi(Ta, Tb) \leq \alpha_1 \varphi(a, b) + \alpha_2 \frac{[\varphi(a, Ta) + \varphi(b, Tb)]}{2} \\
&\quad + \alpha_3 \frac{[\varphi(a, b) + \varphi(Ta, Tb)]}{2} + \alpha_4 \frac{[\varphi(a, Tb) + \varphi(b, Ta)]}{2s} \\
\varphi(a, b) &\leq \alpha_1 \varphi(a, b) + \alpha_2 \frac{[\varphi(a, a) + \varphi(b, b)]}{2} + \alpha_3 \frac{[\varphi(a, b) + \varphi(a, b)]}{2} + \alpha_4 \frac{[\varphi(a, b) + \varphi(a, b)]}{2s} \\
\varphi(a, b) &\leq \alpha_1 \varphi(a, b) + \alpha_3 \varphi(a, b) + \frac{\alpha_4}{s} \varphi(a, b) \\
\varphi(a, b) &\leq (\alpha_1 + \alpha_3 + \frac{\alpha_4}{s}) \varphi(a, b) \Rightarrow \varphi(a, b) = 0, \text{ since } (\alpha_1 + \alpha_3 + \frac{\alpha_4}{s}) < 1.
\end{aligned}$$

This implies that a and b are not different points but are same. Hence 'a' is a unique fixed point of T.

This completes the proof.

3. Conclusion

In present paper, the concept of cone metric spaces and generalized cone b-metric space on fixed points with an example illustrated. Also proved a result on contractive mapping.

References

- [1] Aydi, H, Monica, Felicia, Bota, M, Karapinar, E and Mitrović, S (2012) "A fixed point theorem for set-valued quasi contractions in b-metric spaces", *Fixed Point Theory and Application* no. 88.
- [2] Achari J. (1983): "Fixed point theorems for a class of mappings on non-Archimedean probabilistic metric spaces", *Mathematica*, 25, 5-9.
- [3] Amit Singh, Sandeep Bhatt, Shruti Charkiyal, (2011) "A unique common fixed point theorem for four maps in non-Archimedean Menger PM-spaces", *Int. Journal of Math. Analysis*, Vol. 5 (15), 705-712.
- [4] Alamgir Khan M, Sumitra (2008) "Common fixed point theorem in 2 N. A. Menger PM space for R-weakly commuting maps of type (P)", *Novi Sad J. Math.* (38) (2), 145-152.
- [5] Alamgir Khan M Sumitra: "A common fixed point theorem in non-Archimedean Menger PM-space", *Novi Sad J. Math.*, Vol. 39 (2), 81-87. 2009.
- [6] Alamgir Khan M and Sumitra: "Common fixed point theorems in non-Archidean Menge PM-space", 5 (1), 1-13, 2010.
- [7] Bota, M, Molnar, A and Varga, C: "On ekeland's variational principle in b-metric spaces", *Fixed Point Theory* no. 2, 21-28, 12 (2011).
- [8] Chugh, Rkumar, V, and Kadian, T.: Some fixed point theorems for multivalued mappings in generalized b-metric spaces, *International Journal of Mathematical Archive*, no. 3, 1198-1210, 3 (2012).
- [9] Czerwik, S: "Contraction mappings in b-metric spaces", *Acta Mathematica et Informatica Universitatis Ostraviensis*, 5-11. 1 (1993).
- [10] Czerwik S., Nonlinear set-valued contraction mappings in b-metric spaces, *Atti. Sem. Mat. Fis. Univ. Modena* 46 (1998), 263-276.
- [11] J. R. Morales and E. Rojas: "Cone metric spaces and fixed point theorems of T –Kannan contractive mappings" arXiv: 0907. 3949v2 [math. FA] 26 Oct. 2009.
- [12] Kir, M, Kiziltunc, H: "On some well-known fixed point theorems in b-metric spaces", *Turkish Journal of Analysis and Number Theory*, 1, no. 1, 13-16. (2013).
- [13] Khan, M. S: "A fixed point theorem for metric spaces", *Rent. Inst. Mat. Univ. Trieste*, Vol. III, Fasc. 10 1 – 4. (1976).
- [14] L. G. Haung and X. Zhang: "Cone metric spaces and fixed point of contractive mapping", *J. Math. Anal.*
- [15] M. Jleli, B. Samet, *The Kannan's fixed point theorem in a cone rectangular metric space*, *J. Nonlinear Sci. Appl.* 2 (3) (2009), 161–167. 2009.
- [16] N. Shioji, T. Suzuki and W. Takahashi: "Contraction mappings, Kannan mappings and metric completeness" *Proceedings of the American Mathematical Society*, vol. 126, No. 10, 3117-3124, October 1998.
- [17] R. George, Hossam A. Nabwey, K. P. Reshma and R. Rajagopalan: "generalized cone b-metric spaces and contraction principles" *MATEMATIQKI VESNIK*, 67, 4 (2015), 246–257, December 2015.
- [18] R. George, B. Fisher, "Some generalized results of fixed points in cone b-metric space", *Math. Moravica* 17 (2) (2013), 39–50., 2013.
- [19] S. Rezapour and R. Hamibarani: "Some notes on the paper Cone metric spaces and fixed point Theorems of contractive mappings" *J. Math. Anal. Appl.*, 345, (2), 719-724, 2008.
- [20] Shi, L, Xu, S: "Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces", *Fixed Point Theory and Applications*, no. 120. 2013.
- [21] S. Moradi: "Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depend on another function" arXiv: 0903, 1577v1 [math. FA].
- [22] T. Suzuki, "Generalized metric spaces do not have compatible topology", *Abstract Appl. Anal.* 2014, Art. ID 458098, 5 pages.
- [23] T. Van An, N. Van Dung, Z. Kadelburg, S. Radenović, "Various generalizations of metric spaces and fixed point theorem"s, *Rev. Real Acad. Cienc. Exac., Fis. Nat., Ser. A, Mat.* 109, 1 (2015), 175–198.
- [24] Z. Kadelburg, S. Radenović, Fixed point results in generalized metric spaces without Hausdorff property, *Math. Sci.* 8: 125 (2014).